Critical behaviour associated with helical order near a Lifshitz point

1977 J. Phys. A: Math. Gen. 10 L249

(http://iopscience.iop.org/0305-4470/10/12/008)

J. Phys. A: Math. Gen., Vol. 10, No. 12, 1977. Printed in Great Britain. © 1977

LETTER TO THE EDITOR

Critical behaviour associated with helical order near a Lifshitz point

David Mukamel
Department of Electronics, The Weizmann Institute of Science, Rehovot, Israel

Received 24 August 1977

Abstract. The wavevector $\boldsymbol{q}$ associated with the helical order varies along the paramagnetic-helical critical line as $|\boldsymbol{q}| \sim |p|^{\beta_k}$, as a Lifshitz point ($T = T_L$, $p = 0$) is approached. Renormalisation group techniques in $d = 4 + \frac{1}{2}m - \epsilon (\epsilon > 0)$ dimensions are used to calculate the critical exponent β_k, associated with an m-fold Lifshitz point, to second order in ϵ. For a n-component order parameter we find

$$\beta_k = \frac{1}{2} + \frac{n+2}{(n+8)^2} \frac{11m^2 + 36m + 16}{48(m+2)} \epsilon^2 + O(\epsilon^3).$$

The critical behaviour associated with a Lifshitz point has been discussed extensively in recent years (see, e.g., Hornreich *et al* 1975a,b, 1977, Nicoll *et al* 1976, 1977, Selke 1977, Villain 1977, Abrahams and Dzyaloshinskii 1977, Mukamel and Luban 1977). Two ordered phases exist in the vicinity of a Lifshitz point: a ferromagnetic phase associated with a wavevector $\boldsymbol{q} = 0$, and a helical phase associated with a wavevector $\boldsymbol{q} \neq 0$. The two ordered phases are separated from the disordered (paramagnetic) phase by a critical line with two branches $T_F(p)$ and $T_H(p)$, which intersect at a Lifshitz point. It has been shown (Hornreich *et al* 1975a), using scaling arguments, that along the helical branch of the critical line, the wavevector $\boldsymbol{q}$ associated with the helical order varies as

$$|\boldsymbol{q}| \sim |p|^{\beta_k}, \tag{1}$$

as a Lifshitz point ($T = T_L$, $p = 0$) is approached. Renormalisation group analysis shows that for an m-fold Lifshitz point, $\beta_k = \frac{1}{2} + O(\epsilon^2)$ where $\epsilon = 4 + \frac{1}{2}m - d > 0$ and d is the dimensionality of the system. An m-fold Lifshitz point is characterised by an instability associated with the absence of quadratic terms of the form q_i^2 in the Landau–Ginzburg–Wilson Hamiltonian for all $i = 1, \ldots, m$, $m \leq d$.

In the present letter we calculate β_k for an m-fold Lifshitz point to second order in ϵ. We first use scaling arguments to show that

$$\beta_k = \nu_{l4}/\phi, \tag{2}$$

where ϕ is the crossover exponent and ν_{l4} is one of the two correlation length exponents associated with the Lifshitz point (Hornreich *et al* 1975a, b). We then show that the crossover exponent ϕ can be written in the form† $\phi = \nu_{l4}(2 - \eta_{l4} - \lambda)$ where η_{l4}, $\lambda = O(\epsilon^2)$. By calculating η_{l4} and λ to $O(\epsilon^2)$ we obtain an expression for the exponent β_k.

† This corrects the expression given by Hornreich *et al* (1977).

To derive the scaling relation (2) we note that according to general scaling theory (see, e.g., Fisher 1973, Pfeuty *et al* 1974), the singular part of the susceptibility $\chi_s(t, p, \boldsymbol{q})$ takes the form

$$\chi_s(t, p, \boldsymbol{q}) \sim |t|^{-\gamma} X\left(\frac{p}{|t|^{\phi}}, \frac{\boldsymbol{q}_\alpha}{|t|^{\nu_{l4}}}, \frac{\boldsymbol{q}_\beta}{|t|^{\nu_{l2}}}\right), \tag{3}$$

in the limit t, p, $\boldsymbol{q}_\alpha$, $\boldsymbol{q}_\beta \to 0$. Here $t = (T - T_L)/T_L$, $\boldsymbol{q}_\alpha = (q_1, \ldots, q_m)$, $\boldsymbol{q}_\beta = (q_{m+1}, \ldots, q_d)$, and the exponents γ, ν_{l4}, ν_{l2} and ϕ are those associated with the Lifshitz point. The susceptibility χ_s diverges along the critical line $T_H(p)$ at a wave-vector $\boldsymbol{q}_\alpha \neq 0$. The critical line $T_H(p)$ is therefore defined by the equation $\chi(x_0, y_0, 0) = \infty$ or equivalently by the two equations $p = x_0|t|^{\phi}$ and $|\boldsymbol{q}_\alpha| = y_0|t|^{\nu_{l4}}$. Along the paramagnetic-helical critical line one therefore has $|\boldsymbol{q}_\alpha| \sim |p|^{\nu_{l4}/\phi}$. Comparing this result with equation (1) yields the scaling relation (2). Note that if $\phi > 1$, the leading term in the expression for $p(t)$ along the helical critical line is not the leading singular term $x_0|t|^{\phi}$ but rather a linear term $p \sim ct$. In this case ϕ should be replaced by 1 in the scaling relation (2). However, since $\phi = \frac{1}{2} + O(\epsilon)$, the scaling relation (2) can be used for sufficiently small ϵ. In the following we calculate the ratio ν_{l4}/ϕ and thus obtain the critical exponent β_k.

Consider the n-component Landau–Ginzburg–Wilson Hamiltonian

$$\mathcal{H} = -\tfrac{1}{2}\int_{\boldsymbol{q}} u_2(\boldsymbol{q})\boldsymbol{S}_{\boldsymbol{q}}\cdot\boldsymbol{S}_{-\boldsymbol{q}} - u\int_{q_i} (\boldsymbol{S}_{\boldsymbol{q}_1}\cdot\boldsymbol{S}_{\boldsymbol{q}_2})(\boldsymbol{S}_{\boldsymbol{q}_3}\cdot\boldsymbol{S}_{\boldsymbol{q}_4})\delta(\textstyle\sum \boldsymbol{q}_i), \tag{4}$$

where $\boldsymbol{S}$ is an n-component vector $(S_1, \ldots, S_n)$ and

$$u_2(\boldsymbol{q}) = r + pq_\alpha^2 + q_\beta^2 + q_\alpha^4. \tag{5}$$

The Hamiltonian (4) together with (5) exhibits an m-fold Lifshitz point. The renormalisation group recursion relation for u_2 in $d = 4 + \frac{1}{2}m - \epsilon\,(\epsilon > 0)$ dimensions is given by (see, e.g., Wilson and Kogut 1974, Fisher 1974)

$$u_2' = a^{4-\eta_{l4}}\Big[u_2\Big(\frac{1}{a}\boldsymbol{q}_\alpha, \frac{1}{b}\boldsymbol{q}_\beta\Big) + 4(n+2)u\int_{\boldsymbol{q}_1} G(\boldsymbol{q}_1)$$
$$-32(n+2)u^2\int_{\boldsymbol{q}_1,\boldsymbol{q}_2} G(\boldsymbol{q}_1)G(\boldsymbol{q}_2)G\Big(\boldsymbol{q}_1 + \boldsymbol{q}_2 + \Big(\frac{1}{a}\boldsymbol{q}_\alpha, \frac{1}{b}\boldsymbol{q}_\beta\Big)\Big) + O(u^3, \epsilon^3)\Big], \tag{6}$$

where

$$G(r, p, \boldsymbol{q}) \equiv G(\boldsymbol{q}) = (r + pq_\alpha^2 + q_\beta^2 + q_\alpha^4)^{-1}, \tag{7}$$

$\boldsymbol{q} = (\boldsymbol{q}_\alpha, \boldsymbol{q}_\beta) = (q_1, \ldots, q_m, q_{m+1}, \ldots, q_d)$, and $a(b)$ is the rescale factor associated with the $\boldsymbol{q}_\alpha$ $(\boldsymbol{q}_\beta)$ subspace. The rescale factors a and b satisfy (Hornreich *et al* 1975a)

$$a^{4-\eta_{l4}} = b^{2-\eta_{l2}}. \tag{8}$$

The integrations in (6) extend over the region

$$b^{-(2-\eta_{l2})} < q_{i\alpha}^{4-\eta_{l4}} + q_{i\beta}^{2-\eta_{l2}} < 1, \qquad i = 1, 2.$$

However since we are interested in the exponents to $O(\epsilon^2)$ only, it is sufficient to integrate over the region $b^{-2} < q_{i\alpha}^4 + q_{i\beta}^2 < 1$. Consider now the integral

$$I(\boldsymbol{q}) = \int_{\boldsymbol{q}_1}\int_{\boldsymbol{q}_2} G(\boldsymbol{q}_1)G(\boldsymbol{q}_2)G\Big(\boldsymbol{q}_1 + \boldsymbol{q}_2 + \Big(\frac{1}{a}\boldsymbol{q}_\alpha, \frac{1}{b}\boldsymbol{q}_\beta\Big)\Big), \tag{9}$$

where the propagators G are taken with $r = p = 0$. To calculate the exponent η_{l4} one has to find the $a^{-4} \ln a\, q_\alpha^4$ contribution which comes from the integral $I(\boldsymbol{q})$. Assuming $|\boldsymbol{q}|$ to be small, so that in effect $|\boldsymbol{q}_1 + (a^{-1}\boldsymbol{q}_\alpha, b^{-1}\boldsymbol{q}_\beta)| > b^{-1}$, one can perform the $\boldsymbol{q}_2$ integration. Define first

$$x_i = q_{i\alpha}^2, \qquad y_i = q_{i\beta} \qquad i = 1, 2. \tag{10}$$

In terms of the polar coordinates $(z_i = (x_i^2 + y_i^2)^{1/2}, \theta_i = \tan^{-1}(y_i/x_i))$ $i = 1, 2$, one has

$$\int_{\boldsymbol{q}_2} G(\boldsymbol{q}_2)\left[G\left(\boldsymbol{q}_1 + \boldsymbol{q}_2 + \left(\frac{1}{a}\boldsymbol{q}_\alpha, \frac{1}{b}\boldsymbol{q}_\beta\right)\right) - G(\boldsymbol{q}_1 + \boldsymbol{q}_2)\right]$$
$$\simeq \frac{K_m K_{d-m}}{4} B\left(\frac{m}{4}, \frac{8-m}{4}\right)\left\{\frac{1}{b^2}\left(\frac{1}{z_1^2} - \frac{1}{(q_{1\alpha} + a^{-1}q_\alpha)^4 + (q_{1\beta} + b^{-1}q_\beta)^2}\right)\right.$$
$$\left. + \ln z_1^2 - \ln\left[\left(q_{1\alpha} + \frac{1}{a}q_\alpha\right)^4 + \left(q_{1\beta} + \frac{1}{b}q_\beta\right)^2\right]\right\}, \tag{11}$$

where $B(\alpha, \beta)$ is the usual beta function. Integrating now over $\boldsymbol{q}_1$ and keeping $a^{-4} \ln a\, q_\alpha^4$ terms only we find

$$I(\boldsymbol{q}) - I(0) \sim (K_m K_{d-m})^2 \left(\frac{1}{4} B\left(\frac{m}{4}, \frac{8-m}{4}\right)\right)^2 \frac{1}{24} \frac{m^2+8}{m+2} a^{-4} \ln a\, q_\alpha^4. \tag{12}$$

Using (Hornreich *et al* 1975a, Mukamel and Luban 1977)

$$u^* = \frac{1}{n+8} \frac{1}{K_m K_{d-m} B(m/4, (8-m)/4)} \epsilon^2, \tag{13}$$

we finally obtain

$$\eta_{l4} = -\frac{1}{12} \frac{n+2}{(n+8)^2} \frac{m^2+8}{m+2} \epsilon^2. \tag{14}$$

To calculate the crossover exponent ϕ, one has to linearise the recursion relation for p in the vicinity of the fixed point. The linearised recursion relation takes the form

$$(\Delta p)' = a^{2-\eta_{l4}}[1 - 32(n+2)u^2 A \ln a]\, \Delta p, \tag{15}$$

where $\Delta p = p - p^*$, and A is the coefficient of the $a^{-2} \ln a\, q_\alpha^2$ term which comes from $\partial I(\boldsymbol{q})/\partial p$. Using the same methods we have used to calculate $I(\boldsymbol{q})$ we find

$$A = (K_m K_{d-m})^2 \left(\frac{1}{4} B\left(\frac{m}{4}, \frac{8-m}{4}\right)\right)^2 \frac{m+1}{2}. \tag{16}$$

Exponentiating the recursion relation (15) we obtain

$$(\Delta p)' = a^{2-\eta_{l4}-\lambda} \Delta p, \tag{17}$$

where η_{l4} is given by equation (14), and

$$\lambda = \frac{n+2}{(n+8)^2}(m+1)\epsilon^2. \tag{18}$$

The crossover exponent is thus given by

$$\phi = \nu_{l4}(2 - \eta_{l4} - \lambda), \tag{19}$$

and the exponent β_k is given by

$$\beta_k = \frac{\nu_{l4}}{\phi} = (2 - \eta_{l4} - \lambda)^{-1}. \tag{20}$$

Using (14) and (18) we finally obtain

$$\beta_k = \frac{1}{2} + \frac{n+2}{(n+8)^2} \frac{11m^2 + 36m + 16}{48(m+2)} \epsilon^2 + O(\epsilon^3). \tag{21}$$

Note that because of the assumed spherical symmetry within the $\boldsymbol{q}_\alpha$ and $\boldsymbol{q}_\beta$ subspaces, one expects no helical long-range order for $m \leq d \leq m+1$ (Lubensky 1972, Mukamel and Luban 1977). Therefore, the expression (21) for β_k applies only for $d > m+1$, i.e. for $m < 6$. The exponent β_k has been calculated independently by Hornreich and Bruce (1978) for the special case $m = 1$.

For the cases of most practical interest, namely $m = 1$ and $n = 1, 2$, the exponent β_k is given by

$$\beta_k \simeq 0{\cdot}5 + 0{\cdot}016(4{\cdot}5 - d)^2, \tag{22}$$

and the correction to the classical result is small.

We have also calculated the critical exponent η_{l2}. We find

$$\eta_{l2} = \frac{1}{2} \frac{n+2}{(n+8)^2} \epsilon^2 + O(\epsilon^3), \tag{23}$$

and thus, to second order in ϵ, the exponent η_{l2} is independent of m.

I thank Professor R M Hornreich and Professor A D Bruce for most stimulating correspondence. I also thank Professor M Luban and Professor S Shtrikman for many illuminating discussions. This work was supported in part by a grant from the United States–Israel Binational Science Foundation (BSF), Jerusalem, Israel.

References

Abrahams E and Dzyaloshinskii I E 1977 to be published

Fisher M E 1973 *Proc. 24th Nobel Symp. on Collective Properties of Physical Systems, Aspenaasgaden, Sweden* eds B Lundgvist and S Lundgvist (Stockholm: Nobel Foundation)

—— 1974 *Rev. Mod. Phys.* **46** 597

Hornreich R M and Bruce A D 1978 *J. Phys. A: Math. Gen.* **11** to be published

Hornreich R M, Luban M and Shtrikman S 1975a *Phys. Rev. Lett.* **35** 1678

—— 1975b *Phys. Lett.* **55A** 269

—— 1977 *Physica* **86A** 465

Lubensky T C 1972 *Phys. Rev. Lett.* **29** 206

Mukamel D and Luban M 1977 *Phys. Rev.* B to be published

Nicoll J F, Tuthill G F, Chang T S and Stanley H E 1976 *Phys. Lett.* **58A** 1

—— 1977 *Physica* B **86–8** 618

Pfeuty P, Jasnow D and Fisher M E 1974 *Phys. Rev.* B **10** 2088

Selke W 1977 *Z. Phys.* B **27** 169

Villain J 1977 *Physica* B **86–8** 631

Wilson K and Kogut J 1974 *Phys. Rep.* **12** 75